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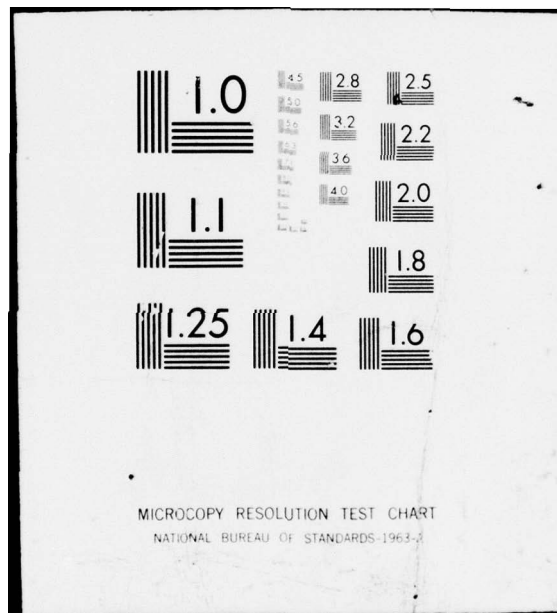
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# ARO Report 77-2

## PROCEEDINGS OF THE TWENTY-SECOND CONFERENCE ON THE DESIGN OF EXPERIMENTS IN ARMY RESEARCH DEVELOPMENT AND TESTING

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ESTIMATE OF RELIABILITY IN THE  
STRESS-STRENGTH MODEL

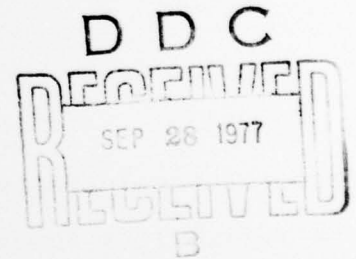
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ABSTRACT

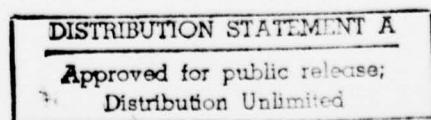
Suppose  $Y$  is the strength of a component which is subject to a stress  $X$ . Then the component fails whenever  $X \geq Y$ , and there is no failure when  $X < Y$ . In this paper the problem of estimating the reliability function

$$R = P(X < Y)$$

is considered. A survey of available results is presented and some new results are considered.



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## INTRODUCTION

Let  $X$  and  $Y$  be two random variables with cumulative distribution functions  $F(x)$  and  $G(y)$  respectively. Suppose  $Y$  is the strength of a component subject to a stress  $X$ . Then the component fails if at any moment the applied stress (or load) is greater than its strength or resistance. The stress is a function of the environment to which the component is subjected, and its value at any point of time is considered a random variable. The strength of a component is measured by the stress required to failure. Strength depends on material properties, manufacturing procedures and so on. If the components under question are mass produced and their selection in a given system is assumed to be made at random, then the strength should also be considered a random variable. The reliability of a component during a given period  $[0, T]$  is taken to be the probability that its strength exceeds the stress during the entire interval, that is, the reliability function  $R$  is given by

$$R = P(X < Y)$$

From practical considerations it is desirable to draw inference about the reliability function. The problem of estimating  $R$  has been considered by many using nonparametric, Bayesian and parametric approach. We shall present a survey of available results and consider some new results.

The above model was first considered by Birnbaum (1956) and has since found an increasing number of applications in many different

areas, especially in the structural and aircraft industries.

As an example, consider the following problem discussed by Lloyd and Lipow (1962). A solid propellant rocket engine is successfully fired provided the chamber pressure  $(X)$  generated by ignition stays below the burst pressure  $(Y)$  of the rocket chamber. If  $X \geq Y$ , the engine blows up and the operation is a failure.

Note the problem of inference about  $R = P(X < Y)$  is similar to the problem of estimation of  $P = P(X \geq Y)$ , the probability of failure. So one can either talk of  $R$ , or of  $P$ .

## 2. Nonparametric approach

Let  $(X_1, X_2, \dots, X_m)$  and  $(Y_1, Y_2, \dots, Y_n)$  be two independent samples of measurements on  $X$  and  $Y$  respectively. Let

$$\phi(X_i, Y_j) = \begin{cases} 1 & \text{if } Y_j < X_i \\ 0, & \text{otherwise} \end{cases}$$

then

$$U = \sum_{i=1}^m \sum_{j=1}^n \phi(X_i, Y_j)$$

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is the well known two sample Mann-Whitney statistic, that is

$$U = \text{number of pairs } (X_i, Y_j) \text{ such that } Y_j < X_i.$$

Birnbaum (1956) showed that the Mann-Whitney statistic  $U$  could be used to estimate  $1 - R$  (Probability of failure), and hence  $R$ . In particular



$$\hat{P} = 1 - \hat{R} = U/mn \quad (2.1)$$

was proposed as an estimator of  $P = \Pr$  (failure), and it was used to obtain one sided confidence interval for  $P$  for the cases  $F$  known,  $G$  unknown ( $m \rightarrow \infty$ ), and both  $F$  and  $G$  unknown. Birnbaum and McCarty (1958) considered a numerical procedure for computing the sample sizes needed for the confidence interval based on  $U/mn$ .

Owen, Craswell and Hanson (1964) showed that the assumption of continuity required in Birnbaum (1956) was not essential and produced some tables for use in computing sample sized and confidence intervals for the Birnbaum-McCarty procedure.

Govindarajulu (1968) also has explicitly derived one sided and two sided distribution free confidence bounds for  $P$  based on the asymptotic normality of  $\hat{P} = U/mn$ . This bounds are approximately one half of the corresponding bounds due to Birnbaum and McCarty (1958). In particular, Govinderajulu showed that for all  $F$  and  $G$  and large  $m$  or  $n$ , the solution  $\epsilon$  of the equations

$$P(P \leq \hat{P} + \epsilon) = P(P \geq \hat{P} - \epsilon) \geq \gamma, \quad 0 < \gamma < 1$$

is given by

$$\epsilon \geq (4v)^{-1/2} \Phi^{-1}(\gamma),$$

and the solution of the equation

$$P(|\hat{P} - P| \leq \epsilon) \geq \gamma, \quad 0 < \gamma < 1$$

is given by

$$\epsilon \geq (4v)^{-1/2} \phi^{-1}\left(\frac{1+\gamma}{2}\right).$$

Here

$$\Phi(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^X e^{-u^2/2} du,$$

and  $\phi^{-1}(\cdot)$  is the inverse function of  $\phi(\cdot)$ .

Recently Govindarajulu (1974) has also considered a sequential distribution-free procedure for obtaining fixed-width confidence limits for  $P$  (and hence for  $R$ ). However, in the absence of additional numerical computation, it is not known how good is the performance of this sequential procedure.

### 3. Bayesian Approach

Not much has been done from the Bayesian point of view. Enis and Geisser (1971) investigated Bayesian approach for estimating  $R$  assuming  $X$  and  $Y$  to be independently distributed and that  $X$  and  $Y$  are either exponentially distributed or normally distributed.

### 4. Parametric Approach

In many situations, the distribution of  $X$  or (of both  $X$  and  $Y$ ) will be known, and it is desired to obtain parametric solutions. Thus, in case of missile flights, the stress may be expensive to sample, but the physical characteristics of the missile system, such as the propulsive force, angle of elevation, changes in atmospheric condition, and so on may all have known distributions; consequently,

the distribution of stresses may be calculated. In this section, we shall consider the problem of estimating  $R$  (or  $P$ ) for specific parametric distributions.

4.1 Normal Distribution: Owen, Craswell and Hanson (1964) considered above problem and gave one sided confidence intervals for  $R$  when both stress and strength are (a) jointly bivariate normally distributed and observations are in pairs, or (b) when  $X$  and  $Y$  are independent normal with a common unknown variance. Note if  $X$  and  $Y$  follow a joint bivariate distribution

$$R = P(X < Y) = P(Y - X > 0) \\ = \Phi \left( \frac{\mu_Y - \mu_X}{\{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2\}^{1/2}} \right)$$

and 
$$\hat{R} = \Phi \{ \bar{Y} - \bar{X} / (\sigma_X^2 - 2\rho\sigma_X + \sigma_Y^2)^{1/2} \}$$

if  $\sigma_X, \sigma_Y$  and  $\rho$  are known. Similarly if  $X$  and  $Y$  are independent

$$P(X < Y) = \int_{-\infty}^{\infty} F(x) dG(x).$$

Same problems have been considered by Govindarajulu (1976), who obtained two sided confidence intervals for  $R$ . Church and Harris (1970) have also considered the same problems under the assumption that  $X$  and  $Y$  are independent, normally distributed and the distribution of  $X$  is known. Assume, without any loss of generality, that  $E(X)=0$  and  $\text{Var}(X)=1$ . In this case,

$$R = P\{X < Y\} = \Phi \left( \frac{\mu}{\sqrt{1+\sigma^2}} \right)$$

where  $\mu = E(Y)$  and  $\sigma^2 = E(Y - \mu)^2$ . Church and Harris considered the estimator

$$R = \phi\left(\frac{\bar{Y}}{\sqrt{1 + s^2}}\right) = \phi(V), \text{ say,}$$

where  $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$  and  $s^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2 / (n - 1)$ , from which they obtained the following confidence interval for  $R$ .

$$P\{\phi(V - \phi^{-1}(1 - \frac{\gamma}{2}) \hat{\sigma}_V) < R < \phi(V + \phi^{-1}(1 - \frac{\gamma}{2}) \hat{\sigma}_V)\} \approx 1 - \gamma$$

Similarly, a one sided confidence interval is given by

$$P\{R > \phi(V - \phi^{-1}(1 - \gamma) \hat{\sigma}_V)\} \approx 1 - \gamma.$$

Here

$$\hat{\sigma}_V = \left[ \frac{s^2}{1 + s^2} \left( \frac{1}{n} + \frac{\bar{Y}^2 s^2}{2(n - 1)(1 + s^2)^2} \right) \right]^{\frac{1}{2}}$$

The confidence interval obtained by Church and Harris compare favorably with that of Govindarajulu (1968). Their procedure, although empirically demonstrated to be superior to that of Govindarajulu is, however, inexact since it uses the asymptotic normal approximation of a given statistic and requires the substitution of the population mean and standard deviations by their observed sample values. In fact, all the parametric estimators suffer from same weakness as

they are based on maximum likelihood estimators. Mazumdar (1970), has considered the same problem of obtaining point and interval estimates of reliability and obtained mvue of reliability using interference theory. Minimum variance unbiased estimator of  $R$  in the normal case has also been considered by Downton (1973).

4.2 Gamma and Exponential distribution: Since in many physical situations, specially in reliability and life testing problems, exponential and gamma distributions provide more realistic models, it is desirable to obtain estimators of  $R$  in these cases.

Let  $X$  and  $Y$  be independently distributed with density functions

$$f(x) = \frac{1}{\Gamma(p)\alpha^p} e^{-x/\alpha} x^{p-1}, \quad x > 0, p > 0$$

$$g(y) = \frac{1}{\Gamma(q)\beta^q} e^{-y/\beta} y^{q-1}, \quad y > 0, q > 0$$

respectively. Then

$$\begin{aligned} R = P(X < Y) &= \int_0^\infty [1 - G(x)] dF(x) \\ &= \int_0^\infty \left[ \int_x^\infty \frac{1}{\Gamma(q)\beta^q} e^{-y/\beta} y^{q-1} dy \right] \frac{1}{\Gamma(p)\alpha^p} e^{-x/\alpha} x^{p-1} dx \\ &= \sum_{k=0}^{q-1} \frac{\Gamma(p+k)}{\Gamma(p)\Gamma(k+1)} \frac{\alpha^k \beta^p}{(\alpha+\beta)^{p+k}}. \end{aligned}$$



Here  $p$  and  $q$  are assumed to be known integers. If two independent random samples  $(X_1, X_2, \dots, X_m)$  and  $(Y_1, Y_2, \dots, Y_n)$  from the two gamma populations are available mle of  $\alpha$  and  $\beta$  are given by  $\hat{\alpha} = \frac{\bar{X}}{p}$  and  $\hat{\beta} = \frac{\bar{Y}}{q}$ . Hence mle of  $R$  is

$$\hat{R} = \sum_{k=0}^{q-1} \frac{\Gamma(p+k)}{\Gamma(p)\Gamma(k+1)} \frac{\hat{\alpha}^k \hat{\beta}^p}{(\hat{\alpha} + \hat{\beta})^{p+k}}.$$

As special cases, if  $q=1$ , that is if  $X$  follows the gamma distribution and  $Y$  follows the exponential distribution

$$\hat{R} = \{\hat{\beta} / (\hat{\alpha} + \hat{\beta})\}^p.$$

Finally, if both  $p$  and  $q$  are equal to 1, we have the case of two independent exponential distributions and we have

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}} = \frac{\bar{Y}}{\bar{X} + \bar{Y}}$$

The distribution of  $\hat{R}$ , for large  $m$  and  $n$ , can be shown to be normal and hence asymptotic confidence interval for  $\hat{R}$  can be obtained.

Tong (1974, 1975) has obtained mvube of  $R$  for gamma and exponential distributions. The variance of the mvube of  $R$ , in the exponential case has been derived by Kelley et al (1976)

**4.3 Weibull distribution:** Let  $X$  and  $Y$  be independent random variables each following the Weibull distribution with common shape



parameter  $\delta$ . That is let

$$F(x) = 1 - e^{-x^\delta/\alpha}, \quad \alpha > 0, x > 0$$

$$G(y) = 1 - e^{-y^\delta/\beta}, \quad \beta > 0, y > 0.$$

We can readily see

$$R = P(X^\delta < Y^\delta) = P(X < Y) = \frac{\beta}{\alpha + \beta}$$

Note above is independent of  $\delta$ . Again, we can obtain the mle of  $R$  to be

$$\hat{R} = \hat{\beta}/(\hat{\alpha} + \hat{\beta})$$

where  $\hat{\alpha}$  and  $\hat{\beta}$  are mle of  $\alpha$  and  $\beta$ .

4.4 Bivariate exponential distribution: Since exponential distribution is considered a useful model in life testing problems, it is desirable to consider bivariate analogue of univariate exponential distributions which will have properties similar to the univariate exponential distribution. Marshall and Olkin (1967) have proposed a very important bivariate exponential distribution (BVE), which is given by

$$\bar{F}(x,y) = P(X > x, Y > y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)}, \quad 0 \leq \lambda_1, \lambda_2, \lambda_{12} < \infty, \lambda_1 + \lambda_{12} > 0, \lambda_2 + \lambda_{12} > 0 (x > 0, y > 0).$$

The BVE does arise in several natural ways and its properties appear to be fundamental. In particular, marginal distributions of BVE are exponential and BVE has the loss of memory property (LMP) given by

$$\bar{F}(x+t, y+t) = \bar{F}(x, y)\bar{F}(t, t) \quad \text{for } s_1, s_2, t \geq 0$$

However, this distribution is not absolutely continuous and there are clearly situations when it can not be applied. Thus, from data, it is found that  $X \neq Y$  for any pair  $(X, Y)$  the model is clearly not applicable. An alternative absolutely continuous distribution related to the BVE and having some of its properties would appear to be of interest. To this end, Block and Basu (1974) have proposed an absolutely continuous bivariate exponential extension (ACBVE), which turns out to be the absolutely continuous part of the BVE of Marshall and Olkin. ACBVE is also seen to be a variant of the distribution Freund (1961). The ACBVE is given by

$$\begin{aligned} \bar{F}(x, y) = & \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)] \\ & - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} \exp[-\lambda \max(x, y)] \quad \text{for } x > 0, y > 0. \end{aligned}$$

Here

$$\lambda = \lambda_1 + \lambda_2 + \lambda_{12} .$$

Estimates of  $R$  when the underlying distribution is BVE or ACBVE has been obtained by Basu (1976). These results will be communicated elsewhere.

##### 5. Reliability of complex systems

The model described before can be extended to more complex systems. For example, a single component system of strength  $Y$  could be subjected to  $k$  different independent stresses  $X_1, X_2, \dots, X_k$ . Here reliability of the system is given by

$$R = P\{X_1 < Y, X_2 < Y, \dots, X_k < Y\}$$

or

$$R = P\{\max(X_1, X_2, \dots, X_k) < Y\}.$$

An example of interest is the case where a beam with strength  $Y$  is subjected to several stresses  $X_1, X_2, \dots, X_k$ . Another similar problem of interest is to evaluate the reliability function  $R'$  of a  $k$ -component system of strengths  $Y_1, Y_2, \dots, Y_k$  respectively each of which is subject to a common stress  $X$ . Here

$$R = P\{X < Y_1, X < Y_2, \dots, X < Y_k\}$$

$$= P\{X < \min(Y_1, \dots, Y_k)\}.$$

As an example, the flow of a current  $X$  through an electronic component assembled from several subcomponents with abilities to accommodate currents  $Y_1, Y_2, \dots, Y_k$  would follow this pattern.

Chandra (1975) has considered the problem of estimating  $R$  and  $R'$  under the assumption that the  $X$ 's and  $Y$ 's are all independent random variables and (a) all follow normal distributions, (b)  $Y$ 's are all exponential and  $X$  is normal with known variance.

Bhattacharyya and Johnson (1974) considered the problem of estimating reliability function  $R$  for a more complex  $m$ -out-of- $k$  system. Here each of  $m$  components of a system of strengths  $Y_1, Y_2, \dots, Y_k$  is subjected to a stress  $X$  and the system survives if at least  $m$  out of the  $k$  components survive. Assuming  $X, Y_1, \dots, Y_k$  to be independent with distribution functions  $F(x), G_1(Y_1), G_2(Y_2), \dots, G_k(Y_k)$ . Bhattacharyya and Johnson considered the problem of estimating the reliability function  $R = P_r(\text{at least } m \text{ of the } Y_1, \dots, Y_k \text{ exceed } X)$ , under the assumption  $G_1 = G_2 = \dots = G_k = G$ , say, and that  $F$  and  $G$  are exponential distributions with known scale parameters. Here

$$R = \sum_{\alpha=m}^k \binom{k}{\alpha} \int_{-\infty}^{\infty} [1-G(x)]^{\alpha} [G(x)]^{k-\alpha} d(F(x)) .$$

Bhattacharyya and Johnson (1973) have also considered a nonparametric approach for the above problem.

The author is currently investigating additional problems in this area results of which will be communicated elsewhere.

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